Denote the greatest common divisor of two moduls $\mathbf{A}_i$ and $\mathbf{A}_j$, that is, $\mathbf{A}_i + \mathbf{A}_j$ by $\mathbf{A}_{ij}$; and write

$$\mathbf{D}_1 = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n, \qquad \mathbf{D}_2 = \mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_1\mathbf{A}_3 + \cdots + \mathbf{A}_{n-1}\mathbf{A}_n,$$
$$\mathbf{D}_3 = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_1\mathbf{A}_2\mathbf{A}_4 + \cdots + \mathbf{A}_{n-2}\mathbf{A}_{n-1}\mathbf{A}_n, \cdots$$
$$\mathbf{D}_{n-1} = \mathbf{A}_2\mathbf{A}_3 \cdots \mathbf{A}_n + \mathbf{A}_1\mathbf{A}_3 \cdots \mathbf{A}_n + \cdots + \mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_{n-1}.$$

Show that

$$\mathbf{A}_{12}\,\mathbf{A}_{13} \cdots \mathbf{A}_{n-1,\,n} = \mathbf{D}_1\mathbf{D}_2 \cdots \mathbf{D}_{n-1}.]$$

2915. Proposed by HARRIS HANCOCK, University of Cincinnati.

Determine x so as to satisfy the two congruences $3x^2 \equiv 0\ (3N)$, $x^3 + a \equiv 0\ (9N^2)$, where $a = N^2 \cdot n$, and the two integers N, n have no common factor, and neither contains a squared factor.

2916. Proposed by HARRIS HANCOCK, University of Cincinnati.

If p is any rational prime integer, and if $\alpha (\neq 1)$ is any root of $x^p - 1 = 0$, show that $p = \mathbf{P}_1 \cdot \mathbf{P}_2 \cdots \mathbf{P}_{p-1}$, where $\mathbf{P}_i$ ($i = 1, 2, \cdots, p - 1$) are the ideals $(p, 1 - \alpha^i)$, which in turn may be reduced to principal ideals. [Remark: This is rather a good elementary example to show that an integer prime in one realm is factorable in a more extended realm.]

2917.

A parabola is rolled upon a fixed right line. Find the locus of (a) its vertex; and (b) its focus.

2918. Proposed by NATHAN ALTSHILLER-COURT, University of Okla.

Find planes which cut four given lines in four concyclic points.

2919. Proposed by V. M. SPUNAR, Chicago, Ill.

An equilateral hyperbola which touches a conic and is concentric with it is called a hyperbolic tangent to the conic. Being given two hyperbolic tangents to a conic, the arc of any third hyperbolic tangent which is intercepted by the first two subtends a constant angle at either focus of the given conic.

NOTES.

19. A Curve of Pursuit. Apropos of Note **10** (*1921*, 184; cf. *1921*, 278, 281), referring to a problem proposed by Lucas in 1877, a very similar problem is given as a worked out exercise in Bateman, *Differential Equations*, 1918, pp. 8–10. It is as follows: "Three boys running at the same speed u chase one another. A pursues B, B pursues C and C pursues A. Find a differential equation which will indicate the way in which the ratios of the sides of the triangle ABC vary." A footnote states that this problem is due to Professor Frank Morley. His differential equation is discussed in a paper by F. E. Hackett, "A numerical solution of the triangular problem of pursuit," in *The Johns Hopkins University Circular*, July, 1908, pp. 135–137.

A. H. WILSON.

20. A Problem in Investment. There has been inquiry concerning the following problem given in a less general form, with a reference to *Engineering News*, volume 48, 1902, pp. 362–363, in E. B. Skinner, *The Mathematical Theory of Investment*, Boston, 1913, p. 140.

Bonds are issued, N in number, of a face value of A each, bearing interest at rate r. At the end of a years and at the end of each year thereafter a certain number of these bonds is to be redeemed at a price which bears the ratio R to the face value A. How many bonds must be redeemed each year in order that the whole issue shall be paid for at the end of n years, and that the sum of the interest

on the unpaid bonds and the amount paid to redeem the bonds shall be the same for each of the $n - a + 1$ years?

This problem is completely solved in *Engineering News* (l.c.).

21. Involutes of a circle and a pasturage problem. On page 128 of *Problems and Solutions. Associateship Examinations, Parts I and II, 1915–1919* (New York, Actuarial Society of America, 1921) is a brief solution of the following problem in the examinations set for 1918: "A circular wall of radius a stands in the middle of a large field. A horse is tethered to the outside of this wall by a rope the length of which is equal to half the circumference of the wall. Show that the area over which the horse can graze is $(5/6)\pi^3a^2$." The area is evidently composed of a semicircle of radius πa and two areas with arcs of involutes of the given circle as outer boundaries.

A somewhat more complicated problem was proposed over one hundred and seventy years ago, in *The Ladies Diary or the Woman's Almanack*, 1748. The problem was as follows (page 41): "Observing a *Horse* tied to feed in a Gentleman's *Park*, with one End of a Rope to his *Fore-foot*, and the other end to one of the Circular *Iron-Rails*, inclosing a Pond, the Circumference of which Rails being 160 yards, equal to the Length of the Rope, what Quantity of Ground, at most could the Horse feed?" A solution was given on pages 25–26 of the *Diary* for 1749, and the answer found was[1] "76257.86 sq. yards = 15*A*.2*R*.12*P*."

Involutes of a circle may be connected with many other curves. For example: If an involute of a circle, of radius a, rolls on a straight line, the locus of the center of the circle is a parabola whose parameter is $2a$ (J. Clerk Maxwell,[2] 1849)—The locus of the center of the circle (radius a) of an involute, rolling on an orthogonal tractory of the catenary, whose equation is

$$y = \frac{x}{2a}\sqrt{x^2 - a^2} + \frac{a}{2}\log\left(\sqrt{\frac{x^2}{a^2} - 1} + \frac{x}{a}\right),$$

is the axis of y (Maxwell, 1849)—If an involute of a circle rolls on an equal involute with corresponding points in contact, the center of the circle traces a spiral of Archimedes (Maxwell, 1849)—The involute of a circle is the locus of the pole of a logarithmic spiral rolling on a concentric circle (Maxwell, 1849; often attributed to Haton de la Goupillière)—The pedal of an involute with respect to the center of its circle is a spiral of Archimedes (Practically the same as the third of the results by Maxwell; see also Mannheim,[3] 1858)—The caustic by reflection of an involute for rays emanating from the center of its circle is an evolute of a spiral of Archimedes (Haton de la Goupillière,[4] 1863)—The inverse of

[1] See also *The Mathematical Questions proposed in the Ladies' Diary*, edited by T. Leybourn, volume 2, London, 1817, pp. 6–7; *The Diarian Miscellany* by C. Hutton, volume 2, London, 1775, p. 269; *The Diarian Repository*, London, 1774, pp. 507–508.

[2] This and the next three results are taken from the remarkable memoir on "The theory of rolling curves" presented to the Royal Society of Edinburgh when Maxwell was 18 years of age. See the *Scientific Papers of James Clerk Maxwell*, volume 1, 1890, pp. xi, 22, 26, 28.

[3] *Nouvelles Annales de Mathématiques*, 1858, pp. 186–187 and 1860, 186–187.

[4] *Nouvelles Annales de Mathématiques*, 1863, pp. 336, 494–500, 548–550.

an involute with respect to the center of its circle is a spiral tractrix, that is, a curve which, in polar coördinates, has a tangent of constant length (Haton de la Goupillière,[4] 1863)—The centers of curvature for the points of contact of an involute rolling on a straight line is a parabola (Cesàro,[1] 1884)—The points of contact of tangents drawn to an involute from any point of its plane lie on a limaçon of Pascal (Fouret,[2] 1888).

The involute of a circle seems to have been first conceived in 1693 when Huygens was considering clocks without pendulums which might be of service on sea going vessels.[3] In this connection he originated an apparatus in which the involute of a circle plays an essential rôle.

In 1891 it became desirable to install in the Royal Observatory at Greenwich a larger telescope. This necessitated that a larger dome, 36 feet in diameter, be built upon a circular wall 31 feet 4 inches in diameter. The form adopted was that of a "surface generated by the revolution of an involute of a circle, 7 feet in diameter, with its center in the plane of the rail, and 5 feet from the axis, the curve being completed near the apex by an arc of a circle (of 13 feet 3 inches radius) so that it cuts the axis at right angles. The diameter of the dome is 36 feet at a height of 7 feet above the rail."[4] ARC.

SOLUTIONS.

2809 [1920, 80]. Proposed by the late L. G. WELD.

Find the nth term of the series defined by the relation, $u_{i+2} = u_i + u_{i+1}$, in which $u_1 = u_2 = 1$.

SOLUTION BY HENRI SEBBAN, Boufarik, Algeria.

Consider the sequence

$$1, 1, 2, 3, 5, 8, \cdots$$

in which each term beginning with the third is the sum of the two terms immediately preceding. In order to express a term as a function of u_1, u_2 and i, let us set $u_i = Ax_1^i + Bx_2^i$ and endeavor to determine A, B, x_1 and x_2 so as to satisfy the law of the sequence. Then

$$\begin{aligned} Ax_1^i + Bx_2^i &= Ax_1^{i-1} + Bx_2^{i-1} + Ax_1^{i-2} + Bx_2^{i-2} \\ &= Ax_1^{i-2}(x_1 + 1) + Bx_2^{i-2}(x_2 + 1), \end{aligned}$$

and from this it follows that $x_1 + 1 = x_1^2$, $x_2 + 1 = x_2^2$. Hence x_1 and x_2 are the two roots of the equation $x^2 - x - 1 = 0$, for we cannot have $x_1 = x_2$ unless the sequence reduces to a geometric progression, $u_i = Kx^i$ of ratio x which is not the case when $u_1 = u_2 = 1$. It will be seen that in the general case A and B are determined by the initial conditions and are different from zero. Hence we may set

$$x_1 = \frac{1 + \sqrt{5}}{2}, \qquad x_2 = \frac{1 - \sqrt{5}}{2}, \qquad \text{and} \qquad u_i = A\left(\frac{1 + \sqrt{5}}{2}\right)^i + B\left(\frac{1 - \sqrt{5}}{2}\right)^i,$$

where A and B are to be determined from the conditions $Ax_1 + Bx_2 = 1$, $Ax_1^2 + Bx_2^2 = 1$. Hence, we have after certain reductions

$$A = -B = \frac{1}{\sqrt{5}} \qquad \text{and} \qquad u_i = \frac{(1 + \sqrt{5})^i - (1 - \sqrt{5})^i}{2^i\sqrt{5}}.$$

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[1] *Mathesis*, 1884, pp. 233–235.

[2] *Journal de Mathématiques Spéciales*, 1888, p. 261.

[3] *Œuvres Complètes de Christian Huygens*, volume 10, 1905, pp. 514–515.

[4] *Monthly Notices of the Royal Astronomical Society*, volume 51, May 8, 1891, p. 436.